Will the real impulse $\delta(t)$ please step $\int^t \delta(t)dt$ up?

Jont B. Allen, ECE
University of Illinois, Urbana IL
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Abstract

I seems to be widely believed that the Fourier and Laplace transforms are simply related to each other. Nothing could be further from the truth. The Fourier transform is the basis for the Hilbert vector-space expansion of signals. The Laplace transform is the basis of system functions, that are causal. The Fourier transform does not naturally include the step function, which must be shoe-horned into the theory. The Laplace transform naturally includes the step function, and thus the delta function, with out the need to introduce *distributions*. In fact we argue that the theory of distributions may be fundamentally flawed.

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1 Introduction

There are important distinctions between the Fourier $\mathscr{F}$ the Laplace $\mathscr{L}$ Transforms that seem to not be widely recognized. The purpose of this note is to raise these distinctions for discussion. Such *integral methods* are widely used for solving differential equations, and it is important recognize the right tool for a specific job.

The definition of the FT pair $f(t) \leftrightarrow F(\omega)$ is

$$F(\omega) = \mathscr{F} f(t) \equiv \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad \leftrightarrow \quad \tilde{f}(t) = \mathscr{F}^{-1} F(\omega) \equiv \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega. \tag{1}$$

The FT takes complex functions on the real line $t : [-\infty, \infty]$ to complex functions on the real line $\omega : [\infty, \infty]$. Commonly the time functions are real, resulting in frequency domain *conjugate symmetry* $F(\omega) = F^*(-\omega)$. As is well known, $\tilde{f}(t) \approx f(t)$ in the L^2 sense, being different only at isolated points on t (i.e., measure zero).

The LT pair $\mathsf{f}(t) \leftrightarrow \mathsf{F}(s)$ is defined as

$$\mathsf{F}(s) = \mathscr{L}\mathsf{f}(t) \equiv \int_{t=0^-}^{\infty} \mathsf{f}(t) e^{-st} dt \quad \leftrightarrow \quad \mathsf{f}(t) = \mathscr{L}^{-1}\mathsf{F}(s) \equiv \frac{1}{j2\pi} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \mathsf{F}(s) e^{st} ds. \tag{2}$$

The LT takes *causal* complex functions $\mathsf{f}(t) \equiv f(t)\mathsf{u}(t)$, where $\mathsf{u}(t)$ is the *Heaviside unit step function* defined on the positive real line $t : [0^-, \infty]$ into complex functions $F(s)$ of complex frequency $s = \sigma + j\omega$, *analytic* over specific regions of the complex s plane. The regions (i.e., points) where the function is not analytic typically correspond to the roots of the characteristic function of the differential equation of some system.

Signals are typically represented by the FT whereas *causal systems*, characterized by a differential equation, are represented by the LT. This is an important distinction we shall further discuss here. As an example of their utility, Laplace transform pairs are essential for describing impedance.

Each pair of integral transforms has certain convergence properties that must be carefully identified. When an integral does not converge it is not defined. These rules are not adequately discussed in the Engineering literature, but will be reviewed here. And each transform method has limitations that make them special. It is the intent of this report to discuss some of these limitations.

Integral methods are fundamental to solving differential equations, and fundamental to any signal processing defined by such differential equations. This is because of the FT convolution theorem

$$f(t) \star g(t) \equiv \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \quad \leftrightarrow \quad F(\omega)G(\omega)$$

and the corresponding LT convolution theorem between causal $\mathsf{f}(t)$ and $\mathsf{g}(t)$ (both zero for negative time)

$$\mathsf{g}(t) \star \mathsf{f}(t) \equiv \int_{0}^{t} \mathsf{f}(t-\tau)\mathsf{g}(\tau)d\tau \quad \leftrightarrow \quad \mathsf{G}(s)\mathsf{F}(s)$$

where $\star$ defines convolution. A useful third convolutional relationship may be defined between a *signal* $f(t) \leftrightarrow F(\omega)$ and a *system* $\mathsf{g}(t)\mathsf{u}(t) \leftrightarrow G(s)$ as

$$\mathsf{g}(t) \star f(t) \equiv \int_{\infty}^{t} \mathsf{g}(t-\tau)f(\tau)d\tau \quad \leftrightarrow \quad F(\omega)\ \mathsf{G}(s)|_{s=j\omega}\,.$$

Alternate forms for such integral relationships also exist, as will be discussed below.

Such convolutional relations frequently exist even when the Fourier transform does not exist. An important example of this is discussed next.

1.1 Limitations of integral transforms

LT limitations: A well know limitation of the LT is limited to causal functions (more generally: one-sided in real-valued time). For example the Heaviside step function corresponds to integration, as indicated by the relationship $\mathsf{u}(t) \leftrightarrow 1/s$, which follows directly from the Cauchy Integral Theorem. It is easily shown that

$$t\mathsf{u}(t) = \mathsf{u}(t) \star \mathsf{u}(t) \leftrightarrow \frac{1}{s^2}, \tag{3}$$

which is a useful and alternative way of expressing $t\mathsf{u}(t) = \int_{\tau=-\infty}^{\infty} \mathsf{u}(t-\tau)\mathsf{u}(\tau)d\tau = \mathsf{u}(t)\int_{\tau=0}^{t} d\tau$. The LT is useless for non-causal signals, but naturally may represent systems, and other causal signals. It may be generalized to alternative one-sided signals. Such representations are referred to as the *2-sided Laplace transform*, which is not really 2-sided, rather it is the some of two 1-sided signals. We shall not need to deal with such generalizations here.

FT limitations: This convolution (Eq. 3) may *not* be computed via the FT, seriously limiting the FT's utility. This is easily seen from the definition of the FT step

$$\hat{u}(t) \equiv \frac{1+\text{sgn}(t)}{2} \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega} = \hat{U}(\omega). \tag{4}$$

When we try to form $\hat{u}(t) \star \hat{u}(t) \leftrightarrow \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)^2$ we see that the integral does not exist, due to the fact that $\delta^2(\omega) \leftrightarrow 1 \star 1$ is not defined. While this is not a startling result, it is significant.

In fact, the FT is that it does not include the step function of the LT, which is undefined at $t = 0$. The FT step, defined as in Eq. 4, is 1/2 at $t = 0$, thus is not equal to the LT step $\mathsf{u}(t)$. Furthermore, due to "Gibbs the phenomena" the inverse Fourier transform of $\hat{U}(\omega)$ does not converge point-wise to $\hat{u}(t)$. The step function of the LT does not have such restrictions.

In terms of the mathematics, the complex plane seems to be special. By wrapping the complex plane around the Riemann sphere, the complex plane becomes *compact*. It may be this compactness, of the Riemann sphere, that naturally separates the Laplace and Fourier Transforms. By being compact, the true step function, without the Gibbs phenomena, is naturally included in the Laplace transform.

Functions having such $\delta(\cdot)$ character are denoted as *finite-power* signals, examples being 1, $\sin(2\pi t)$, $e^{j2\pi t}$, etc. Such constructions are more intuitive extensions of the FT than properly defined mathematics, and widely adopted by the engineering community (but not in mathematics).

In fact, with a slight bit of reflection, it is clear that *all* FT finite-power functions (those containing $\delta(\cdot)$ functions) have related fundamental issues. Because the delta function is not analytic (i.e., one cannot write a Taylor series), it cannot be used in in analytic functions. This is a very fundamental constraint. It is not clear to the author, just how well this point is appreciated by the research community. The construction $\delta^2(\cdot)$ is the simplest example.

What seems interesting here is that while the step function is in this class of finite-power signals, the Heaviside unit step is not in this class, as it has a rigorous mathematical definition

$$\mathsf{u}(t) = \frac{1}{j2\pi}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{1}{s} e^{st} ds,$$

($\sigma_0 > 0$) by using the *Cauchy Integral theorem*. This follows from well known arguments on the convergence (existence) of the integral, which depend critically on the convergence at $s = \infty$,[1] and more specifically, on *closing* the integral at $s \to \infty$, as a function of the sign of real-valued time t.

For the integral to converge (exist) $\Re st = \sigma t$ must be negative as $\omega \to \infty$. Assuming that $\sigma_0 > 0$, for $t > 0$, $\sigma < 0$ (since $\sigma t < 0$), thus the integral must be closed at $\omega \to \infty$ in the left-half s plane, thereby including the pole at $s = 0$. By the same reasoning, for $t < 0$ closure must be in the right-half plane ($\sigma > 0$), thereby evaluating to 0. Thus for $\sigma_0 > 0$, $\mathsf{u}(t) \leftrightarrow 1/s$. Note that $\mathsf{u}(0)$ is not defined by this procedure.

Thus the Laplace transform may be thought of (i.e., it is) an integral representations of analytic functions of real-valued time, in the complex plane. All causal (one sided in time)

[1] http://en.wikipedia.org/wiki/Riemann_surface

functions must be analytic in a half space in the s domain. The step function is the mother of these functions.

The two step functions $\mathsf{u}(t)$ and $\hat{u}(t) \equiv [1 + \text{sgn}(t)]/2$ are distinct, the former being a rigorously defined function via the LT, undefined at $t = 0$, while the latter is in the FT class of finite-power signals with a value of 0.5 at $t = 0$. This seems to the author as a non-trivial point, especially when viewed from the context of solutions to differential equations, and the evaluation of their solution in terms of convolution integrals.

It seems reasonable to define the $\delta(t)$ as

$$\delta(t) \equiv \frac{d\mathsf{u}(t)}{dt} \leftrightarrow \frac{s}{s} = 1 \tag{5}$$

since

$$\mathsf{u}(t) = \int_{\tau=-\infty}^{t} \delta(\tau)d\tau. \tag{6}$$

The first relation is required to represent an inductor, the second a capacitor. These definitions are well defined, consistent with common usage, and appear rigorous.

Finally

$$\delta(t) \star \delta(t) \leftrightarrow 1. \tag{7}$$

represents a cascade of wires of zero length ($\delta(t - L/c)$ represents wire of length L, where c is the wave speed). This last relationship is problematic when one attempts to define it via distributions, as required for the FT case.[2] Feb 2012

I've been telling my students that the delta function (and thus the step function) is a notation, not a function. In mathematics they have defined distribution theory as a way of treating such notions. For me this is artificial, given the true Laplace step function. Unlike $\hat{u}(t)$, $\mathsf{u}(t)$ has no Gibbs ringing and well defined as a function through the inverse Laplace analytic integral relationship $\mathsf{u}(t) \equiv Ł 1/s$. It seems to me that this definition is sufficient to call the Laplace step $\mathsf{u}(t)$ a *function*, but one must allow functions to be undefined at a point (i.e., $t = 0$). More interesting is the question of $\delta(t) = \int_{-\infty}^{t} \mathsf{u}(t)dt$. Is this then also a function, in the same sense? For the moment, the answer is beyond my reach.

References

[2] http://en.wikipedia.org/wiki/Distribution_(mathematics)

Comments and correction history:

Feb 13, 2012 Added a few lines in the conclusion.

Nov 12, 2010 Added distinction re FT vs. LT step functions.

Feb 28, 2010 First pass

1.02 3/2/10 Minor wording changes; integral methods vs. Transform methods.

1.1 3/3/10 More changes, fixing actual mistakes in equations; improved clarity.